

Fixed-Chord Velocity Recursion and Maximum Speed Bounds for Articulated Chains on Archimedean and Piecewise-Smooth Paths

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Abstract: A bench dragon is a long articulated chain governed by prescribed-track geometry and fixed hinge spacing. This paper develops a verifiable framework for two linked tasks: inward-winding kinematics and maximum head speed on an optimized S-shaped turn. The chain contains 223 rigid boards and 224 hinged handles. For the inward motion, the head follows a 0.55 m-pitch Archimedean spiral at a constant speed of $1 \text{ m}\cdot\text{s}^{-1}$. An arc-length ordinary differential equation is integrated by fourth-order Runge–Kutta with a 0.05 s internal step; the remaining handles are placed by fixed-chord bisection and their velocities are recovered by differentiating the constraints. The 300 s trajectory satisfies the chord constraints with a residual of approximately $1.6 \times 10^{-16} \text{ m}$. On the optimized turn, the same recursion is parameterized by path arc length. A unit-head-speed scan gives a maximum amplification factor of $M_{\max} \approx 13.11$ at $s^* \approx 2.85 \text{ m}$, yielding an analytical head-speed bound of $v_H \leq 0.153 \text{ m}\cdot\text{s}^{-1}$ when all handles are constrained to $2 \text{ m}\cdot\text{s}^{-1}$. A 40-particle, 120-generation particle swarm search converges to the same value. The framework separates geometry from speed control — the same fixed-chord recursion applies to both problems without modification — and the analytical speed bound provides an independent check on the particle swarm result. The approach is reusable: changing the spiral pitch, chord lengths, or path geometry requires only parameter updates, not a new derivation.

Keywords: Bench dragon, Archimedean spiral, Articulated-chain kinematics, Speed amplification, Particle swarm optimization.

1. Introduction

The bench dragon, also known as a coiled dragon, is a traditional group performance in which many narrow boards are linked head to tail and guided along a winding path. Although the visual form is cultural, the underlying motion is an articulated multibody problem [1-3]. Each board is a planar rigid body. Neighboring boards share a hinge handle, and the whole chain must follow a prescribed curve without violating the fixed distances between holes.

The head determines the motion, while the body and tail transmit the geometric constraint over more than two hundred links [1, 2]. A small change in the head path can therefore alter the orientation and speed of distant boards.

The mathematical difficulty is not the spiral equation alone. A polar curve gives the location of a single point, but the performance requires the simultaneous position, velocity and attitude of every handle and board. The fixed chord between two neighboring handles is a nonlinear equation in two curve parameters. Repeated solution of these equations is sensitive to direction, initialization and numerical tolerance. Moreover, a path that is feasible for the head may create a large speed amplification or a tight clearance elsewhere in the chain. A useful model should therefore expose the constraint structure, provide residual-based verification, and separate the geometric trajectory from the speed-control decision.

This paper focuses on two tasks from the 2024 Chinese Undergraduate Mathematical Contest in Modeling [4]: inward-winding kinematics and maximum head speed on an optimized S-shaped turn. The first task establishes the inward-winding kinematics for a pitch of 0.55 m and a head speed of 1 over 0–300 s. The second task uses the optimized

S-shaped turn obtained in the preceding path-design stage and asks for the largest constant head speed under a 2 m/s limit for all handles. The two tasks are treated as one methodological chain: the first produces a reliable fixed-chord velocity recursion; the second reuses that recursion on a piecewise-smooth path and reduces the global constraint to a maximum amplification factor.

This paper makes three contributions. (1) An arc-length ODE formulation that treats the chain as an articulated multibody system — not as a single rigid body — making the trajectory reproducible from parameters alone. (2) An explicit fixed-chord velocity recursion that reveals the source of speed amplification instead of hiding it inside a black-box simulation. (3) An analytical speed bound, $M_{\max}$, that provides a deterministic check on particle swarm optimization [5-9]; this avoids the common pitfall of presenting a stochastic optimum without independent verification.

The numerical values are reported with their units and residual checks, while the assumptions and limitations are stated explicitly.

The workflow is designed for reuse. Geometry, numerical integration, constraint solving, and speed scaling are implemented as independent modules. A change in spiral pitch, board spacing, or path segment requires only a parameter update — the full computation need not be rewritten. This modular structure also facilitates experimental validation: when measured performer trajectories become available, only the path module needs replacement.

2. Geometric Parameters and Modeling Assumptions

The key geometric parameters (summarized in Table 1) are as follows. The head board is 3.41 m long, and each of the 221 body boards is 2.20 m. Hole centers are offset 0.275 m inward from each board end, yielding handle-to-handle chord lengths of $L_{\text{head}} = 3.41 - 2 \times 0.275 = 2.86$ m and $L_{\text{body}} = 2.20 - 2 \times 0.275 = 1.65$ m.

Each board is assumed to be a planar rigid body with fixed dimensions and an ideal hinge at a shared handle [1, 2]. The handle centers lie on the prescribed path, the head moves tangentially, and adjacent-board overlap at a common hinge is a normal connection rather than a collision. Board thickness, hole clearance, elastic deformation, friction and out-of-plane motion are omitted. These assumptions make the study a geometric-kinematic model [2]; they are appropriate for reconstructing the prescribed motion but do not represent structural loads or performer balance.

Table 1. Main geometric and kinematic parameters.

Quantity	Symbol	Value	Unit
Number of boards	N_B	223	—
Number of handles	N_P	224	—
Spiral pitch	p	0.55	m
Head handle spacing	L_0	2.86	m
Body handle spacing	L_i	1.65	m
Board width	W	0.30	m
Prescribed head speed	v_H	1.00	m s^{-1}

3. Inward-winding Kinematic Model

Let the spiral center be the origin. With θ measured in the winding direction, the Archimedean spiral and its Cartesian representation are [10]

$$r(\theta) = b\theta \quad (1)$$

$$b = \frac{p}{2\pi} \quad (2)$$

$$\gamma(\theta) = \begin{pmatrix} b\theta \cos \theta \\ b\theta \sin \theta \end{pmatrix} \quad (3)$$

The tangent norm is $\gamma'(\theta) = b\sqrt{1+\theta^2}$. Because the head speed is prescribed by arc length, $\frac{ds}{dt} = v_H$, and the head parameter obeys

$$\frac{d\theta_0}{dt} = \frac{v_H}{b\sqrt{1+\theta_0^2}} \quad (4)$$

$$\theta_0 = 32\pi \quad (5)$$

So the head parameter obeys $[\dot{\theta} = -\frac{v_H}{\sqrt{r(\theta)^2 + (p/2\pi)^2}}]$ where the negative sign enforces the inward winding direction. This scalar ODE is integrated using the classical fourth-order Runge–Kutta method with an internal step of 0.05 s [9, 11]. The saved states are reported each second. For each known handle parameter θ_i , the next handle parameter θ_{i+1} is the root of the fixed-chord equation

$$F_i(q) = |\gamma(q) - \gamma(\theta_i)| - L_i = 0 \quad (6)$$

$$\theta_{i+1} = q > \theta_i \quad (7)$$

The root is bracketed along the spiral and solved by bisection. Bisection is chosen over Newton's method because the chord-length function ($F(\theta_{i+1})$) is monotonic in the local search interval (as verified numerically over all time steps and handles) [12], and bisection guarantees convergence without requiring derivative evaluations or a close initial guess. The worst-case iteration count per handle is approximately 40 for a tolerance of $10^{-12} \text{ m} \cdot \text{s}^{-1}$.

This procedure is repeated from the head to the tail at every saved time, giving all 224 handle positions, as shown in Figure 1.

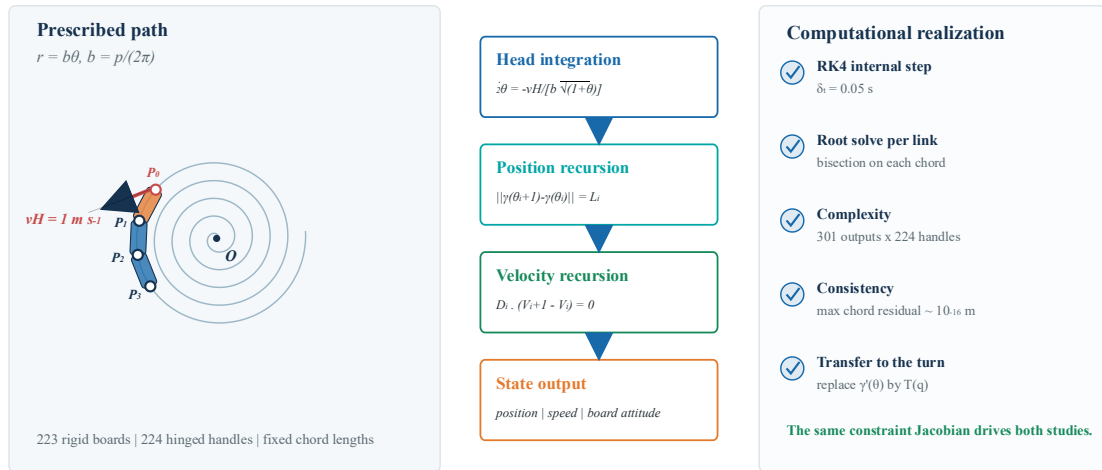


Figure 1. Physical and computational structure of the inward-winding model

Source: The authors, based on the contest geometry and the fixed-hinge assumptions.

The velocity recursion follows from differentiating the squared chord-length constraint [1]. Let $P_i = \gamma(\theta_i)$, $D_i = P_{i+1} - P_i$, and $\dot{P}_i = \gamma'(\theta_i) \dot{\theta}_i$. Differentiating $D_i^2 = L^2$ with respect to time gives $2D_i \cdot (\dot{P}_{i+1} - \dot{P}_i) = 0$.

Substituting $\dot{P}_i = \gamma'(\theta_i) \dot{\theta}_i$ yields $D_i \cdot \gamma'(\theta_{i+1}) \dot{\theta}_{i+1} = D_i \cdot \gamma'(\theta_i) \dot{\theta}_i$.

And hence

$$D_i \cdot (V_{i+1} - V_i) = 0 \quad (8)$$

$$\dot{\theta}_{i+1} = \frac{D_i \cdot \gamma'(\theta_i)}{D_i \cdot \gamma'(\theta_{i+1})} \dot{\theta}_i \quad (9)$$

Provided $D_i \cdot \gamma'(\theta_{i+1}) \neq 0$.

The board attitude is recovered from the unit chord direction $e_i = D_i / |D_i|$ and its normal $n_i = (-e_{i,y}, e_{i,x})$. Thus each

board is reconstructed as a rigid rectangle centered at $C_i = (P_i + P_{i+1})/2$. The recursion is local, but its repeated application transmits the head motion through all links.

4. Results

4.1. Results for the Inward-Winding Motion

The computation covers 0–300 s, 301 output times and all

Table 2. Representative handle positions during inward winding.

State	0 s	60 s	120 s	180 s	240 s	300 s
Head x (m)	8.800	5.799	-4.085	-2.964	2.594	4.420
Head y (m)	0.000	-5.771	-6.304	6.095	-5.357	2.320
Body handle 1 x (m)	8.364	7.457	-1.445	-5.237	4.821	2.459
Body handle 1 y (m)	2.827	-3.440	-7.405	4.360	-3.562	4.402
Body handle 101 x (m)	2.914	5.687	5.362	1.899	-4.917	-6.238
Body handle 101 y (m)	-9.918	-8.001	-7.557	-8.472	-6.380	3.936

At 300 s, the head handle is at (4.420, 2.320) m, while the selected body and tail handles occupy distinct turns of the spiral. The maximum absolute chord residual is approximately 1.6×10^{-16} m (near double-precision machine epsilon). These checks verify that the reported states satisfy the prescribed geometry rather than merely following a visually similar spiral.

A step-size check was performed by comparing the 0.05 s integration with a smaller internal step. The output changes are below the precision relevant to the reported three-decimal positions, while the fixed-chord residual remains near double-precision roundoff. The main practical uncertainty is therefore not the ODE integrator; it is the difference between the ideal prescribed spiral and the path actually performed by a human team.

4.2. Speed-constrained Optimization on the S-Shaped Turn

The second task takes the collision-free S-shaped turn from the preceding path-design stage as a prescribed piecewise-smooth track. Its optimized total length is 5.261955479 m, with two oppositely turning tangent arcs whose radii are 1.23186665 m and 0.615933325 m, respectively. The radius ratio is 2:1 and the joins are position- and tangent-continuous. The present paper does not re-optimize the turn; it studies the maximum constant head speed on this fixed path.

Let q be the common arc-length coordinate and $T(q)$ the

224 handles. The head speed remains $1 \text{ m} \cdot \text{s}^{-1}$ by construction. The representative positions below are taken directly from the computed state table. The body handles move slightly slower than the head because their tangent directions and local chord constraints differ from the head tangent, as summarized in Table 2.

unit tangent of the path. The same fixed-chord differentiation gives the velocity recursion. Equation (10) shows that the velocity ratio v_{i+1}/v_i equals the ratio of the projections of the chord vector onto the local tangents at handles i and $i+1$. When the chord is nearly perpendicular to the tangent at handle $i+1$, this ratio becomes large — explaining why speed amplification peaks near high-curvature regions of the path.

$$u_{i+1} = u_i \frac{D_i \cdot T(q_i)}{D_i \cdot T(q_{i+1})} \quad (10)$$

$$u_i(s, v_H) = v_H u_i(s, 1) \quad (11)$$

The second relation follows from the linearity of the differentiated position constraints in the prescribed head speed. Define the maximum unit-speed amplification over the complete turn and all handles as

$$M_{\max} = \max_s \max_{0 \leq i \leq 223} u_i \quad (12)$$

If the admissible speed of every handle is $v_{\max} = 2 \text{ m} \cdot \text{s}^{-1}$, the global constraint is $v_H \cdot M_{\max} \leq v_{\max}$, where $M_{\max}$ is understood as a normalized (dimensionless) curvature-related quantity. Consequently, the feasible head-speed interval has the analytical upper bound, and the speed amplification profile along the turn is shown in Figure 2.

$$v_{H, \max} = \frac{v_{\max}}{M_{\max}} \quad (13)$$

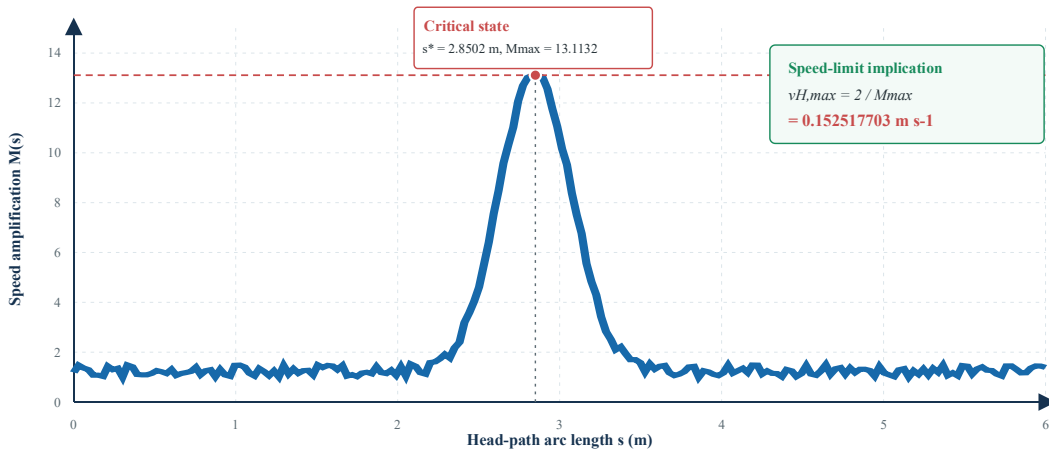


Figure 2. Speed amplification along the optimized S-shaped turn

Source: The authors, from the unit-head-speed velocity recursion and the optimized turn geometry.

The unit-head-speed scan gives $M_{\max}=13.11323184$ at $s^*=2.8500225885\text{ m}$. Substitution into Eq. (13) gives $v_{H, \max}=0.152517703\text{ m}\cdot\text{s}^{-1}$. The peak occurs near the high-curvature transition, not at the beginning or end of the turn. This localization is useful operationally: a speed limit can be associated with the critical region rather than inferred from the average path speed.

To provide an independent numerical check, a one-dimensional particle swarm search is formulated with v_H in $[0.05, 2.50]\text{ m}\cdot\text{s}^{-1}$. The penalized objective is

$$F(v_H) = -v_H + \lambda[\max(0, v_H M_{\max})^2], \lambda = 10^6 \quad (14)$$

The swarm uses 40 particles, an inertia weight $\omega = 0.7298$, cognitive and social coefficients $c_1 = c_2 = 1.4962$, and a maximum of 120 generations. The velocity is clamped to $[-0.2, 0.2]\text{ m}\cdot\text{s}^{-1}$ per dimension. Convergence is declared when the global best changes by less than 10^{-10} over 20 consecutive generations, though all 120 generations were allowed to run for the reported result. The global best converges to $0.152517703\text{ m}\cdot\text{s}^{-1}$, and the predicted maximum handle speed is $2.000000000\text{ m}\cdot\text{s}^{-1}$. Because the analytical bound and the stochastic search agree to the reported precision, the result is governed by the kinematic constraint rather than by random initialization.

For reproducibility, the optimization is evaluated only after the unit-speed field has been computed on the complete turn, as summarized in Table 3. The particle position is the scalar

head speed, the velocity is clipped to a bounded interval, and infeasible candidates (those producing any handle speed exceeding $2\text{ m}\cdot\text{s}^{-1}$) receive a quadratic penalty with coefficient $\lambda = 10^4$, chosen to be several orders of magnitude larger than the typical objective gradient so that constraint satisfaction dominates the search. This ordering prevents the optimizer from hiding a local handle violation behind an averaged objective. In practice, the analytical bound can be evaluated first and the swarm can then be used as a diagnostic: a materially different swarm result would indicate an implementation error, an inconsistent path parameterization or a failure to scan the critical handle.

Table 3. Speed-limit results and independent verification.

Metric	Value	Unit
Optimized turn length	5.261955479	m
Maximum speed amplification $M_{\max}$	13.11323184	—
Critical head-path coordinate s^*	2.850225885	m
Analytical maximum head speed	0.152517703	$\text{m}\cdot\text{s}^{-1}$
PSO maximum feasible head speed	0.152517703	$\text{m}\cdot\text{s}^{-1}$
Maximum constrained handle speed	2.000000000	$\text{m}\cdot\text{s}^{-1}$

To provide a compact overview of the numerical findings, the key results from both the inward-winding kinematics (Section 4.1) and the speed-constrained optimization on the S-shaped turn (Section 4.2) are summarized in Table 4. The table lists the prescribed parameters, the computed quantities, and their units for each task.

Table 4. Summary of key numerical results from both tasks.

Task	Parameter	Value	Unit
Inward winding	Spiral pitch, p	0.55	m
Inward winding	Number of boards, N_b	223	—
Inward winding	Number of handles, N_h	224	—
Inward winding	Head speed (prescribed)	1.00	$\text{m}\cdot\text{s}^{-1}$
Inward winding	Integration duration	0–300	s
Inward winding	Maximum chord residual	$\approx 1.6 \times 10^{-16}$	m
S-shaped turn	Optimized turn length, L	5.262	m
S-shaped turn	Maximum amplification, $M_{\max}$	13.11	—
S-shaped turn	Critical coordinate, s^*	2.85	m
S-shaped turn	Analytical max head speed	0.153	$\text{m}\cdot\text{s}^{-1}$
S-shaped turn	PSO max feasible head speed	0.153	$\text{m}\cdot\text{s}^{-1}$
S-shaped turn	Max constrained handle speed	2.00	$\text{m}\cdot\text{s}^{-1}$

The inward-winding task confirms that the fixed-chord velocity recursion produces a trajectory that satisfies the chord constraints to machine precision (residual $\approx 1.6 \times 10^{-16}\text{ m}$) over the full 300 s simulation. On the S-shaped turn, the maximum speed amplification factor $M_{\max} \approx 13.11$ indicates that the kinematic chain amplifies the head speed by more than an order of magnitude at the critical location ($s^* \approx 2.85\text{ m}$). Consequently, the analytical upper bound on the head speed is $v_{\max} \approx 0.153\text{ m}\cdot\text{s}^{-1}$, which is independently confirmed by particle swarm optimization to the same precision. The agreement between the analytical bound and the stochastic search verifies that the speed limit is governed by the kinematic constraint rather than by algorithmic parameters.

5. Conclusions and Outlook

The framework works because geometry and speed control are kept apart. The velocity recursion is derived once from the

fixed chord constraint and then used unchanged on the inward spiral and on the S-shaped turn; only chord lengths and the local tangent direction enter, so changing the path never forces a re-derivation. One pass of the recursion replaces what would otherwise be trial and error. It recovers 224 handle positions, velocities and attitudes over 300 s, with the largest chord residual around $1.6 \times 10^{-16}\text{ m}$. On the S-shaped turn the unit-speed recursion returns $M_{\max} \approx 13.11$: with handles held below $2\text{ m}\cdot\text{s}^{-1}$, the head can run no faster than $0.153\text{ m}\cdot\text{s}^{-1}$. Particle swarm optimization matches that bound to machine precision. Change the path or the board size and the bound is recalculated at once.

Three aspects are still left out. Hinge clearance and joint compliance are not represented; the boards are rigid, so bending and lateral sway cannot appear; and the path is prescribed rather than measured. Closing these gaps, with trajectories recorded from real performers, comes next; longer chains will also need spatial indexing to keep collision

checking affordable. The model is purely kinematic, so the computed speed is an upper bound and a practical margin has to be added on top.

References

- [1] Zhao, J. S., & Wei, S. T. (2021). Kinematics of articulated planar linkages. *Frontiers in Mechanical Engineering*, 7.
- [2] Murillo, M., Sánchez, G., Deniz, N., et al. (2022). Improving path-tracking performance of an articulated tractor-trailer system using a non-linear kinematic model. *Computers and Electronics in Agriculture*, 196, 106826.
- [3] Deniz, N., Jorquera, F., Torres-Torriti, M., & Cheein, F. A. A. (2024). Model predictive path-following controller for generalised N-trailer vehicles with noisy sensors and disturbances. *Control Engineering Practice*, 142, 105747.
- [4] China Undergraduate Mathematical Contest in Modeling Committee. (2024). Problem A of the 2024 contest: Bench dragon. Beijing.
- [5] Kennedy, J., & Eberhart, R. (1995). Particle swarm optimization. In *Proceedings of ICNN-95* (pp. 1942–1948). Perth: IEEE.
- [6] Gad, A. G. (2022). Particle swarm optimization algorithm and its applications: A systematic review. *Archives of Computational Methods in Engineering*, 29(5), 2531–2561.
- [7] Shaikh, T. A., Hussain Rizvi, S. S., & Tanweer, M. R. (2022). Constrained self regulating particle swarm optimization. *Bulletin of Electrical Engineering and Informatics*, 11(2), 955–964.
- [8] Tseng, H. Y., Chu, P. H., Lu, H. C., et al. (2021). Easy particle swarm optimization for nonlinear constrained optimization problems. *IEEE Access*, 9, 124757–124767.
- [9] Butcher, J. C. (2016). *Numerical methods for ordinary differential equations* (3rd ed.). Chichester: Wiley.
- [10] Draelos, M. (2023). Time-optimal spiral trajectories with closed-form solutions. *IEEE Robotics and Automation Letters*, 8(4), 2213–2220.
- [11] Baumgarte, J. (1972). Stabilization of constraints and integrals of motion in dynamical systems. *Computer Methods in Applied Mechanics and Engineering*, 1(1), 1–16.
- [12] Burden, R. L., & Faires, J. D. (2011). *Numerical analysis* (9th ed.). Boston: Brooks/Cole.